Find two linearly independent series solutions of $2y'' + xy' + 3y = 0$ about $x = 0$.

SCORE: _____ / 15 PTS

+ 2 BONUS POINTS

1 EACH UNLESS OTHERWISE NOTED

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1)a_{n+2} + (n+3)a_n]x^n = 0$$

$$4a_2 + 3a_0 = 0 \quad \Rightarrow \quad a_2 = -\tfrac{3}{4}a_0$$

$$2(n+2)(n+1)a_{n+2} + (n+3)a_n = 0 \quad \Rightarrow \quad a_{n+2} = -\frac{n+3}{2(n+2)(n+1)}a_n \text{ for } n \in Z^+$$

NOTE: recurrence relation also applies if $n = 0$ ie. $a_2 = -\frac{3}{2(2)(1)}a_0 = -\frac{3}{4}a_0$

$a_0 = 1, \quad a_1 = 0$ 1/2

$a_2 = -\frac{3}{2(2)(1)}a_0 = -\frac{3}{2(2)(1)}$ 1/2

$a_3 = a_5 = a_7 = \ldots = 0$

$a_4 = -\frac{5}{2(4)(3)}a_2 = \frac{5(3)}{2^2(4)(3)(2)(1)}$ 1/2

$a_6 = -\frac{7}{2(6)(5)}a_4 = -\frac{7(5)(3)}{2^3(6)(5)(4)(3)(2)(1)}$ 1/2

$$y_1 = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)(2k-1)(2k-3)\ldots(5)(3)}{2^k(2k)!}x^{2k}$$

$$y_1 = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)(2k)(2k-1)(2k-2)(2k-3)\ldots(5)(4)(3)(2)}{2^k(2k)!(2k)(2k-2)\ldots(4)(2)}x^{2k}$$

$$y_1 = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)!}{2^k(2k)!2^k(k)(k-1)\ldots(2)(1)}x^{2k}$$

$$y_1 = \sum_{k=0}^{\infty} (-1)^k \frac{2k+1}{2^{2k}k!}x^{2k}$$

1 BONUS

$a_0 = 0, \quad a_1 = 1$ 1/2

$a_2 = a_4 = a_6 = \ldots = 0$

$a_3 = -\frac{4}{2(3)(2)}a_1 = -\frac{4}{2(3)(2)}$ 1/2

$a_5 = -\frac{6}{2(5)(4)}a_2 = \frac{6(4)}{2^2(5)(4)(3)(2)}$ 1/2

$a_7 = -\frac{8}{2(7)(6)}a_5 = -\frac{8(6)(4)}{2^3(7)(6)(5)(4)(3)(2)}$ 1/2

$$y_2 = x + \sum_{k=1}^{\infty} (-1)^k \frac{(2k+2)(2k)(2k-2)\ldots(6)(4)}{2^k(2k+1)!}x^{2k+1}$$

$$y_2 = x + \sum_{k=1}^{\infty} (-1)^k \frac{2^k(k+1)(k)(k-1)\ldots(3)(2)}{2^k(2k+1)!}x^{2k+1}$$

$$y_1 = x + \sum_{k=1}^{\infty} (-1)^k \frac{(k+1)!}{(2k+1)!}x^{2k+1}$$

$$y_1 = \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)!}{(2k+1)!}x^{2k+1}$$

1 BONUS

Find two linearly independent series solutions of $2x^2y'' - xy' + (x+1)y = 0$ about $x = 0$. **SCORE: _____ / 15 PTS**

+2 BONUS POINTS

1/2 EACH UNLESS OTHERWISE NOTED

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} + (x+1)\sum_{n=0}^{\infty}a_nx^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r} - \sum_{n=0}^{\infty}(n+r)a_nx^{n+r} + \sum_{n=0}^{\infty}a_nx^{n+r+1} + \sum_{n=0}^{\infty}a_nx^{n+r} = 0 \quad 1$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r} - \sum_{n=0}^{\infty}(n+r)a_nx^{n+r} + \sum_{n=1}^{\infty}a_{n-1}x^{n+r} + \sum_{n=0}^{\infty}a_nx^{n+r} = 0$$

$$2r(r-1)a_0x^r - ra_0x^r + a_0x^r + \sum_{n=1}^{\infty}[(2(n+r)(n+r-1)-(n+r)+1)a_n + a_{n-1}]x^{n+r} = 0$$

$$1 \quad (2r(r-1)-r+1)a_0x^r + \sum_{n=1}^{\infty}[(2(n+r)(n+r-1)-(n+r)+1)a_n + a_{n-1}]x^{n+r} = 0$$

$$2r(r-1)-r+1 = 0 \quad \Rightarrow \quad (2r-1)(r-1) = 0 \quad \Rightarrow \quad r = 1, \tfrac{1}{2} \quad 1$$

$$(2(n+r)(n+r-1)-(n+r)+1)a_n + a_{n-1} = 0 \quad \Rightarrow \quad a_n = -\tfrac{1}{2(n+r)(n+r-1)-(n+r)+1}a_{n-1} = -\tfrac{1}{(2(n+r)-1)(n+r-1)}a_{n-1} = -\tfrac{1}{(2n+2r-1)(n+r-1)}a_{n-1}$$

$r = 1 \quad \Rightarrow \quad a_n = -\tfrac{1}{(2n+1)n}a_{n-1}$

$a_0 = 1$

$a_1 = -\tfrac{1}{(3)(1)}a_0 = -\tfrac{1}{(3)(1)}$

$a_2 = -\tfrac{1}{(5)(2)}a_1 = \tfrac{1}{(5)(3)(2)(1)}$

$a_3 = -\tfrac{1}{(7)(3)}a_2 = -\tfrac{1}{(7)(5)(3)(3)(2)(1)}$

$1 \quad y_1 = x(1 - \tfrac{1}{(3)(1)}x + \tfrac{1}{(5)(3)(2)(1)}x^2 - \tfrac{1}{(7)(5)(3)(3)(2)(1)}x^3 + \ldots)$

$y_1 = x(1 - \tfrac{2}{(3)2(1)}x + \tfrac{2\cdot 2}{(5)(3)(4)(2)}x^2 - \tfrac{2\cdot 2\cdot 2}{(7)(5)(3)(6)(4)(2)}x^3 + \ldots)$

$y_1 = \sum_{n=0}^{\infty}(-1)^n\tfrac{2^n}{(2n+1)!}x^{n+1} = \sum_{n=1}^{\infty}(-1)^{n+1}\tfrac{2^{n-1}}{(2n-1)!}x^n$

1 BONUS

$r = \tfrac{1}{2} \quad \Rightarrow \quad a_n = -\tfrac{1}{2n(n-\frac{1}{2})}a_{n-1} = -\tfrac{1}{n(2n-1)}a_{n-1}$

$a_0 = 1$

$a_1 = -\tfrac{1}{(1)(1)}a_0 = -\tfrac{1}{(1)(1)}$

$a_2 = -\tfrac{1}{(2)(3)}a_1 = \tfrac{1}{(2)(1)(3)(1)}$

$a_3 = -\tfrac{1}{(3)(5)}a_2 = -\tfrac{1}{(3)(2)(1)(5)(3)(1)}$

$1 \quad y_2 = x^{\frac{1}{2}}(1 - \tfrac{1}{(1)(1)}x + \tfrac{1}{(2)(1)(3)(1)}x^2 - \tfrac{1}{(3)(2)(1)(5)(3)(1)}x^3 + \ldots)$

$y_2 = x^{\frac{1}{2}}(1 - \tfrac{2}{(2)(1)}x + \tfrac{2\cdot 2}{(4)(2)(3)(1)}x^2 - \tfrac{2\cdot 2\cdot 2}{(6)(4)(2)(5)(3)(1)}x^3 + \ldots)$

$y_2 = x^{\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n\tfrac{2^n}{(2n)!}x^n$

1 BONUS